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# IMPROVING THE CONVERGENCE OF THE MODAL DECOMPOSITION OF INTERNAL WAVES GENERATED BY A MOVING DIPOLE†

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Existing results on the singularities in the neighbourhood of the plane of symmetry of the wave track of the fundamental solution of the linear equation of internal waves in Boussinesq form, obtained for exponential stratification are extended to cover arbitrary stratification. Use is made of well-known asymptotic expansions in powers of the mode number of the eigenvalues and eigenfunctions of the Sturm–Liouville problem. Apart from the principal singularity, available when using the method of "frozen coefficients", the next singularity is singled out and it is demonstrated that this is an essential part of the approximate calculation of the wave pattern near the plane of symmetry of the track. © 1999 Elsevier Science Ltd. All rights reserved.

In existing solutions [1, 2] of similar problems of the perturbation field of a fluid is represented as an expansion in terms of internal wave modes. Two regions in which the corresponding series are irregular are the near field and the neighbourhood of the plane of symmetry of the wave track. Constructive representations of the solution in these regions have been obtained for the special case of uniform stratification [3],‡ and the principal term of the near-field singularity has been singled out for any Väisälä–Brunt frequency distribution [4]. The results reported in the paper cited in the footnote that apply to both regions of irregularity are extended to a fluid of variable stratification with one maximum of the Väisälä–Brunt frequency.

## 1. STATEMENT OF THE PROBLEM

Consider the perturbation field generated by a submerged dipole in uniform motion in a layer of inviscid incompressible vertically-stratified fluid.

Suppose the fluid occupies the region $-\infty < x_1, y < \infty, -h < z < 0$, and the Väisälä–Brunt frequency $N(z)$ depends only on the vertical coordinate $z$. At a depth $h_0$ from the top of the fluid $z = 0$ a point dipole with moment $M$, oriented in the direction of motion, is moving with constant velocity $c$ in the negative direction of the horizontal axis $x_1$. In the system of coordinates associated with the dipole $x = x_1 - ct$, using a Boussinesq approximation and the "solid cover" condition on the surface $z = 0$, the steady field of vertical displacements of fluid particles $\zeta(x, y, z)$ is described in a linear formulation by the equation with boundary conditions

$$\Delta\zeta_{xx} + N^2(z)c^{-2}\Delta_2\zeta = Mc^{-1}\delta(x, y, z + h_0)_{xxz}, \quad \zeta(-h) = \zeta(0) = 0 \tag{1.1}$$

$$(\Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial y^2, \quad \Delta = \Delta_2 + \partial^2/\partial z^2)$$

to which must be added the radiation condition: basic wave perturbations are formed behind the dipole.

An exact solution of problem (1.1) is obtained in [2] in the form of single integrals of an expansion in powers of the modes ($H(\cdot)$ is the Heaviside function)

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$$\zeta = M(2\pi c)^{-1}[\zeta_0(|x|, y, z) + H(x)\zeta_1(x, y, z)] \tag{1.2}$$

$$\zeta_0 = \operatorname{Im} \int_0^{i\infty} \sum_{n=1}^{\infty} \varphi_n \exp(-|\beta_n^{1/2}|\mu) H(-\beta_n) d\theta \tag{1.3}$$

$$\zeta_1 = -\operatorname{Im} \int_{-\pi/2}^{\pi/2} \sum_{n=1}^{\infty} \varphi_n \exp(i\beta_n^{1/2}\mu) d\theta \tag{1.4}$$

$$\mu = x\cos\theta + y\sin\theta, \quad \lambda = (c\cos\theta)^{-2}$$

$$\varphi_n(z, h_0; \lambda) = w_n(z; \lambda) w_{nz}(-h_0; \lambda) \left( \int_{-h}^{0} w_n^2(z; \lambda) dz \right)^{-1}$$

Expressions (1.3) and (1.4) contain the eigenvalues $\beta_n(\lambda)$, $(\beta_1 < \beta_2 < \beta_3 < \ldots)$ and eigenfunctions $w_n(z; \lambda)$ of the Sturm–Liouville problem

$$w_{zz} + [N^2(z)\lambda - \beta]w = 0 \ (-h < z < 0), \quad w(-h) = w(0) = 0 \tag{1.5}$$

Formula (1.2) separates out two components of the vertical displacement field. The wave numbers in the modal expansion in the first of these ($\zeta_0$) are pure imaginary, so that $\zeta_0$ describes the near field of perturbations of the fluid. In the second ($\zeta_1$) the wave numbers are real and $\zeta_1$ represents the wave track behind the dipole.

The expansion of $\zeta(x, y, z)$ in a sum of modes is most appropriate for calculating the perturbation field in cases where one can confine oneself to a small number of modes. If many modes are taken into account in (1.3) and (1.4) or in the analogous formulae in [1] it is then necessary to establish the corresponding number of dispersion relations $\beta_n(\lambda)$ and eigenfunctions $w_n(z; \lambda)$ over a wide range of variation of the parameter $\lambda$, which involves a large amount of computation in cases of practical importance, when $\beta_n(\lambda)$ and $w_n(z; \lambda)$ cannot be found analytically.

There are two regions of the perturbation field which must be taken into account in this connection. The first of these is the near region $R < h$, $R = \sqrt{(x^2 + y^2)}$, where series (1.3) converges slowly and its terms have a logarithmic singularity at $R = 0$. The second is the neighbourhood of the plane of symmetry of the wave track $y = 0$, which enters the zone of basic wave perturbations of all the modes.

A constructive representation $\zeta(x, y, z)$ for these regions can be obtained by isolating the singularities of the terms of the solution $\zeta_0$ and $\zeta_1$, thereby ensuring the corresponding series to converge more rapidly (cf. the paper cited in the footnote).

## 2. ISOLATION OF THE NEAR-FIELD SINGULARITIES

We base our investigation of series (1.3) on the fact that the values of the parameter $\lambda$ are bounded on the integration contour in (1.3) $(0 \leqslant \lambda \leqslant c^{-2})$.

We use the asymptotic forms of the eigenvalues and eigenfunctions of problem (1.5) as $n \to \infty$

$$\beta_n = -k_n^2, \quad k_n = \frac{\pi n}{h} - \lambda\frac{r(0)}{\pi n} + O\left(\frac{\lambda}{n^2}\right)$$

$$w_n(z; \lambda) = \sin\frac{\pi n z}{h} + \frac{\lambda}{\pi n} q(z) \cos\frac{\pi n z}{h} + O\left(\frac{\lambda}{n^2}\right) \tag{2.1}$$

$$r(z) = \frac{1}{2}\int_{-h}^{z} N^2(z_1) dz_1, \quad q(z) = r(z)h - r(0)(z+h)$$

A sufficient condition for (2.1) to be valid [5] is that the variation of the function $N^2(z)$ should be bounded. Using (2.1), we find the asymptotic form as $n \to \infty$, uniform with respect to the parameter of integration $\theta$, of terms of series (1.3)

$$\varphi_n \exp(-k_n\mu) = b_{n0} + b_{n1} + \varepsilon_n, \quad \varepsilon = O(\lambda|\mu|/n)$$

$$b_{n0} = \pi n B_n f_{n0}, \quad b_{n1} = \lambda B_n [f_{n1} + r(0)\mu f_{n0}]; \quad B_n = \frac{1}{h^2}\exp\left(-\frac{\pi n}{h}\mu\right)$$

$$f_{nk} = f_{nk}^{+} + f_{nk}^{-},\ k = 0,1;\quad f_{n0}^{\pm} = \sin\frac{\pi n}{h}(z \pm h_0),\quad f_{n1}^{\pm} = p_{\mp}(z)\cos\frac{\pi n}{h}(z \pm h_0)$$

$$p_{\pm}(z) = q(z) \pm q(-h_0)$$

We now represent the expansion for $\zeta_0$ in the form

$$\zeta_0 = \zeta_{00} + \zeta_{01} + \zeta_{02} \tag{2.2}$$

$$\zeta_{0k} = \operatorname{Im}\int_0^{i\cdot\infty}\sum_{n=1}^{\infty} b_{nk}d\theta,\quad k = 0,1 \tag{2.3}$$

$$\zeta_{02} = \operatorname{Im}\int_0^{i\cdot\infty}\sum_{n=1}^{\infty}[\varphi_n\exp(-k_n\mu)H(-\beta_n) - b_{n0} - b_{n1}]d\theta \tag{2.4}$$

Note that when $R = 0$ the series in (2.3) diverge and the series in (2.4) no longer has that singularity.

We now put $\zeta_{00}$ and $\zeta_{01}$ in a form which enables us to compute $\zeta_0$ for small values of $R$. To do so, we first use Poisson's summation formula to transform the series

$$\sum_{i=1}^{\infty} b_{n0} = -\frac{\mu}{\pi}\sum_{m=-\infty}^{\infty}\frac{\partial}{\partial z}\left[\frac{1}{\mu^2 + (d_m + h_0)^2} + \frac{1}{\mu^2 + (d_m - h_0)^2}\right]$$

$$d_m = z + 2mh$$

Then, substituting the resulting series into the expression for $\zeta_{00}$ and integrating it term-by-term, we obtain an expression which converges for all $R$

$$\zeta_{00}(x,y,z) = \eta_0(R, z + h_0) + \eta_0(R, z - h_0) \tag{2.5}$$

$$\eta_0(R,z) = \frac{1}{2}\sum_{m=-\infty}^{\infty} d_m(d_m^2 + R^2)^{-3/2}$$

We then derive a similar expression for $\zeta_{01}$, using (2.5). Note first that

$$\frac{\partial^2}{\partial x^2}\zeta_{01} = \frac{1}{c^2 h}\left[p_+\frac{\partial}{\partial z}\eta_0(R, z - h_0) + p_-\frac{\partial}{\partial z}\eta_0(R, z + h_0) - r(0)\left(R\frac{\partial}{\partial R} + 2\right)\zeta_{00}\right]$$

Substituting $\eta_0$ and $\zeta_{00}$ from (2.5) into this formula and then integrating the resulting series twice with respect to $x$, we find

$$\zeta_{01} = p_+\eta_1(x,y,z - h_0) + p_-\eta_1(x,y,z + h_0) + r(0)[\eta_2(x,y,z - h_0) + \eta_2(x,y,z + h_0)] \tag{2.6}$$

$$\eta_\nu = \frac{1}{2c^2 h}\sum_{m=-\infty}^{\infty}\frac{d_m^{\nu-1}}{(y^2 + d_m^2)^2}\left[x(d_m^2 - y^2) - x^2\frac{d_m^2}{\sqrt{R^2 + d_m^2}} + y^2\sqrt{R^2 + d_m^2}\right]$$

The terms with $m = 0$ of the series $\eta_0(R, z)$ and $\eta_\nu(x, y^2)$ ($\nu = 1, 2$) are irregular at the point where the dipole is situated $x = y = 0$, $z = -h_0$—they absorb the singularity of the solution, and the actual series converge everywhere. If there is no stratification (we have a homogeneous fluid, $N^2(z) \equiv 0$) the function $M(2\pi c)^{-1}\zeta_{00}$ is an exact solution of problem (1.1).

Thus, from expression (2.2), in which $\zeta_{00}$ is calculated using formula (2.5), and $\zeta_0$ using formula (2.6), we can calculate the term $\zeta_0$ of the near-field solution of the problem. Note that since the terms of the series in the expression for $\zeta_0$ of (2.2) do not decrease as $n$ increases when $R = 0$, it is not enough just to isolate $\zeta_{00}$ only.

Examples which illustrate how the convergence of $\zeta_0$ improves once the singularity for $N^2(z) = \text{const}$, $y = 0$ has been isolated are given in the paper cited in the footnote. It is found that the above procedure is most effective in the case where the dipole velocity $c$ is greater than the velocity of propagation of long internal waves $c_n$.

## 3. TRANSFORMATION OF THE EXPRESSION FOR THE WAVE TERM

It is difficult to obtain the asymptotic form of terms of the series $\zeta_1$ using formula (1.4) directly, because the parameter $\lambda$ of problem (1.3) is unbounded along the integration contour in (1.4). Thus here we require a different expression for $\zeta_1$, for which we can give a uniform estimate of the terms of the respective series as $n \to \infty$. The double integral from which formula (1.4) was obtained [2] can be written in the form

$$\zeta_1 = \frac{1}{\pi}\operatorname{Im}\int_0^{\pi/2}\left\{\int_{\sigma_-} D\,d\beta - \int_{\sigma_+} D\,d\beta\right\}d\theta \tag{3.1}$$

$$D = -\frac{\partial}{\partial h_0}G(z-h_0;\lambda,\beta)\sin(\beta^{1/2}x\cos\theta)\cos(\beta^{1/2}y\sin\theta),\quad \operatorname{Re}\beta^{1/2} \geqslant 0$$

where $G(z, \xi; \lambda, \beta;)$ is Green's function of problem (1.5), the integration contour $\sigma_-$ goes along the real axis from zero to infinity, avoiding the poles of $G$ from below along small semi-circles in the complex plane of the parameter $\beta$ and contour $\sigma_+$ is the complex conjugate of $\sigma_-$.

Making the replacement of variables

$$\beta = \lambda\tau^2;\quad \lambda = (c\cos\theta)^{-2};\quad d\theta = (2c\lambda)^{-1}(\lambda - c^{-2})^{-1/2}d\lambda$$

we change the order of integration in (3.1). Then, using the fact that $\operatorname{Im}\lambda \cdot \operatorname{Im}\beta \geqslant 0$ on the dispersion curves [2], we reduce the expression for $\zeta_1$ to the form

$$\zeta_1 = \frac{1}{\pi}\operatorname{Im}\int_0^{N_m}\tau\sin\frac{x\tau}{c}\left\{\int_{\omega_-} F\,d\lambda - \int_{\omega_+} F\,d\lambda\right\}d\tau \tag{3.2}$$

$$F = -\frac{\partial G}{\partial h_0}\frac{\cos(y\tau\sqrt{\lambda - c^{-2}})}{\sqrt{\lambda - c^{-2}}}$$

The integration contour $\omega_-$ passes along the real axis of the parameter $\lambda$ from $c^{-2}$ to infinity, going around the poles of $G$ from below and the contour $\omega_+$ is the complex conjugate of $\omega_-$. The variables of integration in (3.2) are the parameters of the problem

$$v_{zz} + \lambda Q(z;\tau)v = 0\ \ (-h < z < 0),\quad v(0) = v(-h) = 0;\ Q = N^2(z) - \tau^2 \tag{3.3}$$

For real $\tau \geqslant N_m = \max N(z)$, problem (3.3) has no positive eigenvalues $\lambda$. Hence, for $\tau \geqslant N_m$ Green's function has no poles in the neighbourhood of the positive part of the real axis $\lambda$. Thus, we can take finite limits from 0 to $N_m$ in the integration with respect to $\tau$ in (3.2). Using the theorem of residues to compute the inner integrals in (3.2), we obtain

$$\zeta_1 = \frac{2}{c}\int_0^{N_m}\tau\sin\frac{x\tau}{c}H(\lambda_n - c^{-2})\sum_{n=1}^{\infty} f_n(y,z,h_0;\tau)\,d\tau \tag{3.4}$$

$$f_n = \psi_n\cos(y\tau\sqrt{\lambda_n - c^{-2}})/\sqrt{\lambda_n - c^{-2}}$$

$$\psi_n(z,h_0;\tau) = v_n(z,\tau)v_{nz}(-h_0;\tau)\left[\int_{-h}^{0} Q(z;\tau)v_n^2(z;\tau)\,dz\right]^{-1}$$

where $\lambda_n(\tau)$ and $\upsilon_n(z;\tau)$ are the eigenvalues and eigenfunctions of problem (3.3).

Expression (3.4) is more convenient than (1.4) because the corresponding problem (1.5) is reduced to the form (3.3).

## 4. ISOLATING THE SINGULARITY OF THE WAVE TERM

The asymptotic form $f_{n0}$ of terms of series (3.4) as $n \to \infty$ is derived from the known asymptotic forms [6] of the eigenvalues $\lambda_n$ and eigenfunctions $\upsilon_n$ of problem (3.3). The asymptotic forms of $\lambda_n$ and $\upsilon_n$ as

$n \to \infty$ depend on the number of turning points—the solutions of the equation $N(z) = \tau$. We shall assume that the function $N(z)$ is sufficiently smooth and is either monotone or has one maximum at the point $z = z_m$, $N_m = N(z_m)$ and $N'(z) \neq 0$ for $z \neq z_m$. Then problem (3.3) has no more than two turning points $z_1(\tau) < z_2(\tau)$ for $0 \leqslant \tau < N_m$ and $Q(z; \tau) > 0$ for $z_1 < z < z_2$. We complete the definition of the function $z_k(\tau)$ $(k = 1, 2)$ by putting $z_1(\tau) = -h$ when $Q(-h; \tau) > 0$ and $z_2(\tau) = 0$ when $Q_0$; $\tau > 0$. Denoting the number of turning points by $\nu$, we can now write the formulae of [6] for the asymptotic forms $\lambda_n$ and $\upsilon_n$ for $\nu = 0$, 1 or 2 in the form

$$\lambda_n^{1/2} = \lambda_{n0}(\tau)/J(\tau) + O(n^{-1}), \quad \lambda_{n0} = \pi n - \nu\pi/4$$

$$\upsilon_n(z;\tau) = Q^{-1/4}\sin[\lambda_{n0}I/J + \varepsilon_\nu\pi/4] + O(n^{-1}), \quad Q(z;\tau) > 0 \tag{4.1}$$

$$\upsilon_{nz}(-h_0;\tau) = Q_0^{1/4}\lambda_{n0}J^{-1}\cos[\lambda_{n0}I_0/J + \varepsilon_\nu\pi/4] + O(1)$$

$$\upsilon_n(z;\tau) = O(n^{-\infty}), \quad \upsilon_{nz}(z;\tau) = O(n^{-\infty}), \quad Q(z;\tau) < 0$$

$$Q_0(\tau) = Q(-h_0,\tau); \quad \varepsilon_0 = 0, \quad \varepsilon_1 = \varepsilon_2 = 1$$

$$I(z,\tau) = \int_{z_1}^{z} \sqrt{Q(z',\tau)}dz', \quad I_0(\tau) = I(-h_0,\tau)$$

$$J = \int_{-h}^{0} Q(z,\tau)\upsilon_n^2(z,\tau)dz = \frac{1}{2}I(z_2,\tau) + O(n^{-1})$$

The asymptotic forms (4.1) for $\lambda_n$ cannot be used when $|\tau - N(-h)| \ll 1$, $|\tau - N(0)| \ll 1$ and $|\tau - N_m| \ll 1$. Moreover, the asymptotic forms of the eigenfunctions are invalid in the neighbourhood of turning points for $|z - z_k(\tau)| \ll 1$. The uniform asymptotic forms of solutions of problem (3.4) can be expressed in terms of Airy functions [6]. It can be shown that the errors arising from the use of non-uniform asymptotic forms in integral (3.4) are of smaller order as $n \to \infty$ than the principal terms of the asymptotic forms. From formulae (4.1) we deduce that as $n \to \infty$

$$f_n \sim f_{n0}$$

$$f_{n0} = \frac{2}{J}\left(\frac{Q_0}{Q}\right)^{1/4}\sin\left[\lambda_{n0}\frac{I}{J} + \varepsilon_\nu\frac{\pi}{4}\right]\cos\left[\lambda_{n0}\frac{I_0}{J} + \varepsilon_\nu\frac{\pi}{4}\right]\cos\left(\lambda_{n0}\frac{y\tau}{J}\right) \tag{4.2}$$

Hence it follows that $f_n = O(1)$ as $n \to \infty$, and so the series

$$S = \sum_{n=1}^{\infty} f_n, \quad S_0 = \sum_{n=1}^{\infty} f_{n0} \tag{4.3}$$

must be considered in a generalized sense.

Substituting (4.2) into (4.3), we first convert the product of the trigonometric functions into sums. Then, using the formulae

$$\sum_{n=1}^{\infty} \sin n\alpha = \frac{1}{2}\operatorname{ctg}\frac{\alpha}{2}, \quad \sum_{n=1}^{\infty} \cos n\alpha = -\frac{1}{2} + \sum_{m=-\infty}^{\infty} \delta\left(\frac{\alpha}{\pi} - 2m\right)$$

we derive the following expressions

$$S_0 = \frac{1}{4J}\left(\frac{Q_0}{Q}\right)^{1/4}\left(\operatorname{ctg}\frac{\pi}{2}D_- + \operatorname{ctg}\frac{\pi}{2}D_+\right), \quad \nu = 0$$

$$S_0 = \frac{1}{2J}\left(\frac{Q_0}{Q}\right)^{1/4}\left\{\frac{1}{4}\left(\sin\frac{\pi}{4}D_-\right)^{-1} + \frac{1}{4}\left(\cos\frac{\pi}{4}D_+\right)^{-1} + \right.$$

$$\left. + \sum_{m=-\infty}^{\infty} (-1)^m[\delta(D_- + 2 - 4m) + \delta(D_+ - 4m)]\right\}, \quad \nu = 1 \tag{4.4}$$

$$S_0 = \frac{1}{2J}\left(\frac{Q_0}{Q}\right)^{1/4}\left[\frac{1}{2}\left(\sin\frac{\pi}{2}D_-\right)^{-1} + \sum_{m=-\infty}^{\infty}(-1)^m\delta(D_+ - 2m)\right], \quad \nu = 2$$

$$D_\pm = (l \pm l_0 - y\tau)/J$$

Thus, we have found the sums of the series $S_0$, that is the singularity function, which has absorbed the singularity of the integrand in (3.4). The formula

$$\zeta_1 = \zeta_{10} + \eta_1 \tag{4.5}$$

$$\zeta_{10} = \frac{2}{c}\int_0^{N_m}\tau\sin\frac{x\tau}{c}S_0 d\tau$$

$$\eta_1 = \frac{2}{c}\sum_{n=1}^{\infty}\left[\int_0^{N_m}\tau\sin\frac{x\tau}{c}H(\lambda_n - c^{-2})f_n d\tau - \int_\Omega \tau\sin\frac{x\tau}{c}f_{n0}d\tau\right]$$

where $\Omega$ is the interval for $\tau$ in which $Q(z', \tau) > 0$, is the sum of terms with an isolated singularity and a convergent series, and can be used to calculate wave perturbations generated by a dipole near the plane $y = 0$.

Examples which show the improvement in convergence of the expansion in powers of modes $\zeta_1$ after the two stages of isolating singularities for $N^2(z)$ = const, together with the perturbation patterns in the plane $y = 0$, are given in the paper cited in the footnote.

Note that the condition on the smoothness of the function $N(z)$ under which the singularity of the term $\zeta_1$ was isolated was stronger than that required for $\zeta_0$. In particular, if the function $N(z)$ is piecewise-constant, which simplifies the calculation of the dispersion dependences of problems (1.5) and (3.3), there are no asymptotic forms (4.1) or asymptotic forms of terms of series (3.4) derived from them.

Analysis of formulae (1.2), (2.2) and (4.5) reveals a relatively simple dependence of the terms $\zeta_0$ and $\zeta_1$ on the dipole velocity $c$. For $c \gg 1$ in the near region the perturbations of the fluid depend only slightly on both the parameter $c$ and on the stratification, and the wave perturbations behind the dipole are of order $c^{-1}$, their longitudinal scale being proportional to $c$ and the transverse scale being slightly dependent on $c$.

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